INTRODUCTION TO MECHANICAL VIBRATIONS

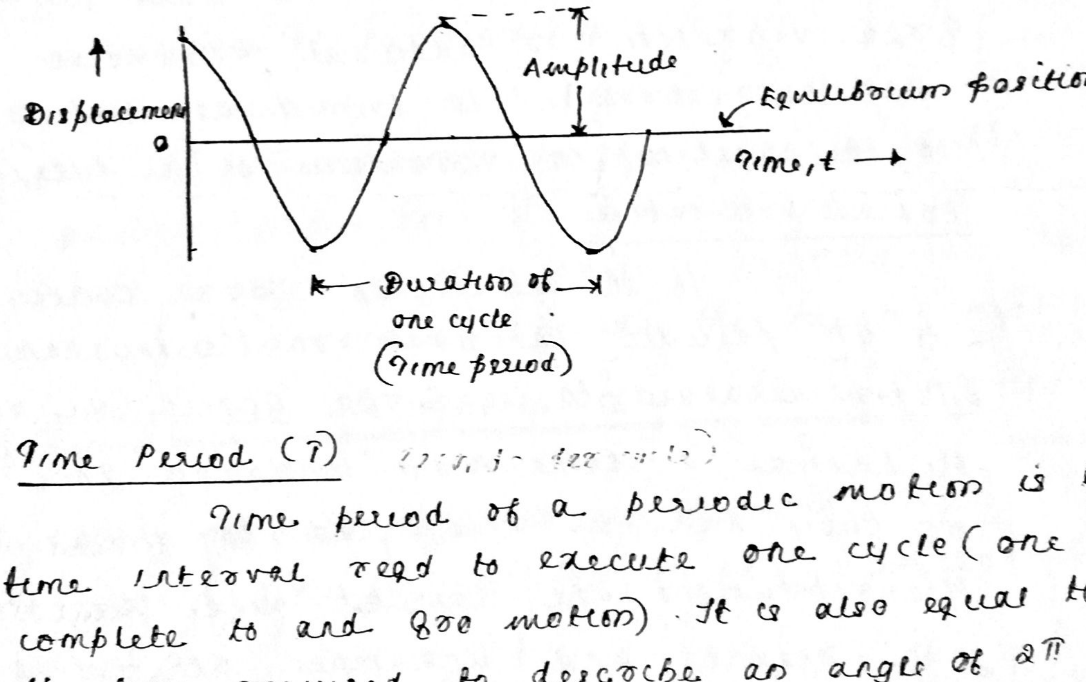
A mechanical vibration is the motion of a particle or a body which oscillates about a position of equilibrium. It results when a system is displaced from a position of stable equilibrium. The system tends to return to this position under the action of restoring force.

Periodic motion / Vibratory motion / Oscillatory motion

A motion which repeats itself in equal intervals of time is called periodic motion.

eg: rise and fall of tides, oscillation of electric circuits, vibrations of strings of a piano etc; guitar etc.

A system undergoing periodic motion in terms of its displacement time plot is shown below



Time Period (T)

Time period of a periodic motion is the time interval reqd to execute one cycle (one complete to and fro motion). It is also equal to the time required to describe an angle of 2π radians. Time taken by a particle for one complete oscillation.

Frequency or cyclic frequency - (f or N, or ν)

It is the no. of cycles executed per unit time and is the inverse of time period

$$\text{frequency}, f \text{ or } N \text{ or } \nu = \frac{1}{T}$$

It is expressed in cycles per second and unit is Hertz

1 hertz = 1 cycle/sec

Angular or circular frequency (ω)

Angular frequency is 2π times the cyclic frequency and is usually expressed in rad/s.

$$\omega = 2\pi N = \frac{2\pi}{T}$$

Amplitude:

Amplitude of a periodic motion is the maximum displacement of the system from its equilibrium position.

Free vibration:

If a disturbing force is applied to the system just to start the vibration and is then removed from the system leaving it to vibrate by itself, then the system is said to undergo free vibration or natural vibration. The frequency of this vibration is known as natural frequency of the system. eg: vibrations of a tuning fork.

Forced vibration

If the disturbing force continues to act at periodic intervals on the system, the system is said to undergo forced vibration.

Mechanical systems may undergo free vibration or they may be subjected to forced vibrations. The vibrations are damped when friction forces are present and undamped otherwise.

SIMPLE HARMONIC MOTION (SHM)

Simple harmonic motion is the simplest type of periodic motion. In SHM, the particles moves to and fro, either along a straight line or along the arc of a circle, about a fixed point called the equilibrium or mean position (when the vibrating body of a system has to and fro motion along a straight line, the vibration is called linear vibration or longitudinal vibration.

Characteristics of SHM:

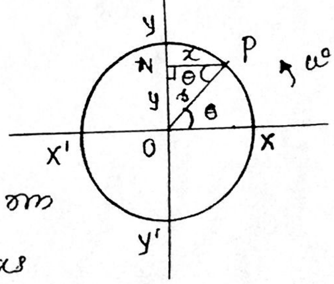
i) The motion is oscillatory about a point lying on the path of the motion, called mean position or position of equilibrium.

ii) The acceleration of a particle executing SHM is always proportional to the distance of the particle from the mean position.

iii) The acceleration of a particle executing SHM is always directed towards the mean position.

Velocity and acceleration of a particle moving with SHM

SHM may be defined as the projection of a uniform circular motion on any diameter of the circle.



Consider a particle moving along the circumference of a circle of radius r with a uniform angular velocity of ω rad/s as shown in figure.

Let P be the position of the particle at some instant after 't' sec from X.

∴ Angle turned by the particle,

$\theta = \omega \times t$ radians

Displacement of point N
(ie, projection of point P on the vertical dia YY' of the circle) $= y = ON = r \sin\theta$

ie, $y = r\sin\omega t \longrightarrow (1)$

$\Rightarrow \sin\omega t = y/r$

where r is the amplitude of SHM.

Differentiating eqn (1) wrt time,

$$\frac{dy}{dt} = r\omega\cos\omega t$$

ie, velocity, $V_y = r\omega\cos\omega t \longrightarrow (2)$

ie, $V_y = r\omega\sqrt{1-\sin^2\omega t}$

$= r\omega\sqrt{1-y^2/r^2} = r\omega\sqrt{\frac{r^2-y^2}{r}}$

$\therefore \boxed{V_y = \omega\sqrt{(r^2-y^2)}}$

Similarly, $\boxed{V_x = -\omega\sqrt{(r^2-x^2)}}$

$\left(x = \ldots \quad \frac{dx}{dt} = -r\omega\ldots \quad V_x = -r\omega\ldots = -r\omega\sqrt{1-x^2/\ldots} \quad V_x = -\omega\sqrt{(r^2-x^2)} \quad a_x = \frac{dV_x}{dt} = -r\omega^2\ldots \quad a_x = -\omega^2 x\right)$

Differentiating eqn (2) w.r.t time,

$$\frac{dv_y}{dt} = \frac{d^2y}{dt^2} = -r\omega^2 \sin\omega t$$

$$= -\omega^2 \times r\sin\omega t$$

ie, $\boxed{a_y = -\omega^2 y}$ where $y = r\sin\omega t$

similarly, $\boxed{a_x = -\omega^2 x}$

NOTE :

1) The minus sign shows that the direction of acceleration is opposite to the direction in which y increases.

ie, the acceleration is always directed toward the point O. But in actual practice, this relation is used as $a = \omega^2 y$.

Maximum and Minimum velocity and acceleration of a particle moving with SHM

Velocity of a particle moving with SHM,

$$v = \omega\sqrt{r^2 - y^2}$$

Velocity is maximum, when $y = 0$ or when N passes through O, the mean position

$\therefore$ Max velocity, $\boxed{V_{max} = r\omega}$

Velocity is zero, when $y = r$, ie, when N passes through y' or y as shown in figure

Acceleration of SHM, $a = \omega^2 y$

Acceleration is maximum, when $y = r$, ie N passes through y or y'

$$\boxed{a_{max} = \omega^2 r}$$

Acceleration is zero, when $y = 0$, ie, when N passes through O, ie, the mean position.

Problems:

1. A body is vibrating with SHM of amplitude 100 mm and frequency 2 vibn/sec. Calculate the max velocity and acceleration?

Amplitude $= r = 0.1\ m$

$N = 2$ vibn/sec

$$V_{max} = r\omega$$
$$= r \cdot 2\pi N$$
$$= 0.1 \times 2\pi \times 2$$
$$= 1.257\ m/s$$

$$a_{max} = \omega^2 r$$
$$= (2\pi N)^2\, r = (2 \times \pi \times 2)^2 \times 0.1 = 15.79\ m/s^2$$

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2. Find the amplitude and time period of a partic moving with SHM, which has a velocity of 9 m/s and 4 m/s at the distance 2m and 3m respectively from the centre? Calculate max velocity & max accln.

$r = ?, \quad T = ?$

$V_2 = 9$ m/s

$V_3 = 4$ m/s.

$V = \omega\sqrt{r^2 - y^2}$

$9 = \omega\sqrt{r^2 - 2^2}$

$\Rightarrow 81 = \omega^2(r^2 - 4) \longrightarrow (1)$

Also, $4 = \omega\sqrt{r^2 - 3^2}$

$\Rightarrow 16 = \omega^2(r^2 - 9) \longrightarrow (2)$

Solving (1) and (2),

$\frac{(1)}{(2)} \Rightarrow \frac{81}{16} = \frac{r^2 - 4}{r^2 - 9}$

$81r^2 - 729 = 16r^2 - 64$

$\Rightarrow r = 3.2\text{m}$

Substituting in (1),

$\omega = 3.6$ rad/s.

Now, $T = \frac{2\pi}{\omega} = 1.74$ s

$V_{max} = r\omega = 3.2 \times 3.6$
$= 11.52$ m/s

$a_{max} = \omega^2 r = 3.6^2 \times 3.2$
$= 41.472$ m/s^2

3. A point moves with SHM. When this point is 0.75 m from midpath, its velocity is 11 m/s and when 2 m from the centre of its path, its velocity is 5 m/s. Find its period and its greatest acceleration?

when $y = 0.75$ m, $V = 11$ m/s.

when $y = 2$ m, $V = 5$ m/s.

$T = ?, \quad a_{max} = ?$

ie, $11 = \omega\sqrt{r^2 - 0.75^2} \longrightarrow (1)$

$5 = \omega\sqrt{r^2 - 2^2} \longrightarrow (2)$

$$\frac{①}{②} \Rightarrow \frac{11}{5} = \frac{\sqrt{r^2 - 0.75^2}}{\sqrt{r^2 - 2^2}}$$

$$\Rightarrow r = 2.21 m$$

Substituting in ①,

$$\Rightarrow \omega = 5.29 \text{ rad/s}$$

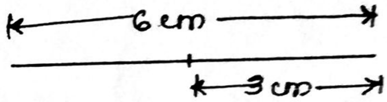
$$T = \frac{2\pi}{\omega} = 1.19 \text{ sec}$$

$$a_{max} = \omega^2 r$$

$$= 5.29^2 \times 2.21$$

$$= 62.55 \text{ m/s}^2$$

4. A particle is executing SHM in a line 6cm long. Its velocity is 12 cm/s while passing through the centre of the line. Find its period.



$$r = \frac{0.06}{2} = 0.03 m$$

when $y = 0$, $V = 0.12$ m/s

ie, $0.12 = \omega\sqrt{r^2 - y^2}$

$$0.12 = \omega\sqrt{(0.03)^2 - 0^2}$$

$$\Rightarrow \omega = \frac{0.12}{0.03} = 4 \text{ rad/s}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.57 \text{ sec}$$

5. A particle follows SHM. Find the position of the particle from the centre when its velocity is 50% of the max velocity.

$$V_{max} = r\omega$$

$$50\% \, V_{max} = 0.5 \times r\omega$$

ie, $V = \omega\sqrt{r^2 - y^2}$

ie, $0.5 r\omega = \omega\sqrt{r^2 - y^2}$

$$0.5^2 r^2 \cancel{\omega^2} = \cancel{\omega^2}(r^2 - y^2)$$

$$y^2 = r^2 - 0.5^2 r^2 = 0.75 r^2 \Rightarrow y = 0.866 r$$

6. A body performing SHM has a velocity of 12 m/s when the displacement is 50 mm and 3 m/s when the displacement is 100 mm, the displacement being measured from the midpoint. Calculate the frequency and amplitude of the motion. What is the acceleration when displacement is 75 mm?

when $y = 0.05$ m, $v = 12$ m/s

$y = 0.1$ m, $v = 3$ m/s.

$N = ?$, $r = ?$, when $y = 0.075$ m, $a = ?$.

i.e.,

$$12 = \omega\sqrt{r^2 - 0.05^2} \longrightarrow (1)$$

$$3 = \omega\sqrt{r^2 - 0.1^2} \longrightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{12}{3} = \sqrt{\frac{r^2 - 0.05^2}{r^2 - 0.1^2}}$$

$$\Rightarrow r = 0.1025 \text{ m}$$

$$\Rightarrow \omega = 134.1117 \text{ rad/s}$$

$$N = \frac{\omega}{2\pi} = \frac{134.1117}{2\pi} = 21.35 \text{ cycles per sec}$$

$$a = \omega^2 y$$

$$= (134.1117)^2 \times 0.075$$

$$= 1348.95 \text{ m/s}^2$$

7. A particle moving with SHM performs 10 complete oscillations per minute and its speed is 60% of the max speed when it is at a distance of 8 cm from the centre of oscillation. Find the amplitude and max acceleration of the particle. Also find the speed of the particle when it is 6 cm far from the centre of oscillation?

$N = 10$ complete cycle per minute

$= \frac{10}{60}$ cycles/sec

$= 0.1666$ cycles/sec

when $y = 0.08\,m$, $V = 60\%\ V_{max}$.

$= 0.6 \times r\omega$

ie, $0.6 r\omega = \omega\sqrt{r^2 - 0.08^2}$

$0.6^2 r^2 \cancel{\omega^2} = \cancel{\omega^2}(r^2 - 0.08^2)$

$0.08^2 = r^2(1 - 0.6^2)$

$\Rightarrow 0.08 = r \times 0.8$

$\Rightarrow r = 0.1\,m \Rightarrow \omega = 2\pi N = 1.04719$ rad/s.

$a_{max} = \omega^2 r$

$= (1.04719)^2 \times 0.1$

$= 0.1096\ m/s^2$

when $y = 0.06\,m$,

$V = \omega\sqrt{r^2 - y^2}$

$= 1.04719\sqrt{0.1^2 - 0.06^2}$

$= 0.0837\ m/s$

8. The frequency of a particle following SHM is 2 cycles per sec. Its speed is 4 m/s at its mean position. Findout the distance between 2 extreme positions. Also find its speed when it is half way b/w mid-position and one extreme position?

Tutorial ①

$N = 2$ cycles/sec.

$V = 4\ m/s$, when $y = 0$

$\omega = 2\pi N$

$= 2\pi \times 2 = 4\pi$ rad/s.

$\therefore V = \omega\sqrt{r^2 - y^2}$

ie, $4 = 4\pi\sqrt{r^2 - 0^2} \Rightarrow 1/\pi = r = 0.3183\ m$

Distance between 2 extreme positions

$= 2r = 2 \times 0.3183\ m$

$= 0.63661\ m$

when $y = r/2$,

$V = \omega\sqrt{r^2 - (r/2)^2}$

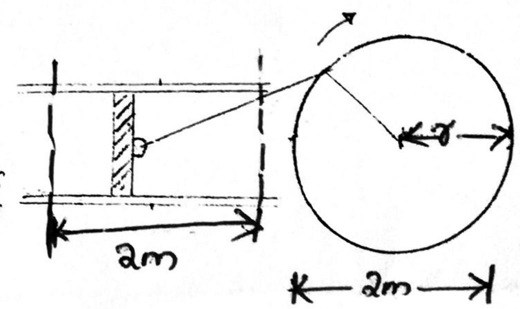
$= 4\pi\sqrt{(0.3183)^2 - (0.3183/2)^2}$

$= 3.48\ m/s$

9. The piston of a steam engine moves with SHM. The crank rotates at 120 rpm and the stroke length is 2m. Find the velocity and acceleration of the piston, when it is at a distance of 0.75m from the centre.

Tutorial 7

$\omega = 120$ rpm



$N = 120$ rpm

$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 120}{60}$ rad/s

$= 12.566$ rad/s

Stroke length $= 2m$

ie, Amplitude, $r = 2/2 = 1m$

when $y = 0.75$,

$V = \omega\sqrt{r^2 - y^2}$

$= 12.5666\sqrt{1^2 - 0.75^2}$

$= 8.31\ m/s$

$a = \omega^2 y$

$= (12.566)^2 \times 0.75$

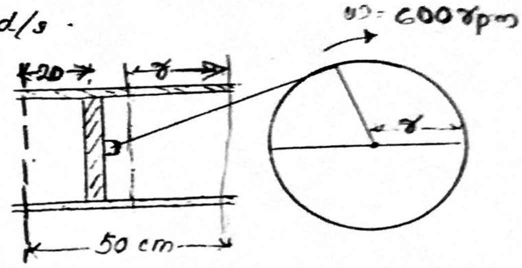
$= 118.428\ m/s^2$

10. The piston of an IC engine follows SHM. The crank rotates at 600 rpm and stroke of the engine is 50 cm. Find the velocity and acceleration of the piston when it is 20 cm, from one of the dead centre positions?

Tutorial 8

$N = 600\ rpm$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 600}{60} = 20\pi\ rad/s$$



Stroke length = 0.5 m

ie, $2r = 0.5\ m$

$r = 0.25\ m$

Here, $x = r - 0.2$

$= 0.25 - 0.2$

$= 0.05\ m$

$V = \omega\sqrt{r^2 - x^2}$

$= 20\pi\sqrt{(0.25)^2 - (0.05)^2}$

$= 15.4\ m/s$

$a = \omega^2 x$

$= (20\pi)^2 \times 0.05$

$= 197.4\ m/s^2$

11\) For a particle undergoing SHM, the displacement is prescribed by the cosine function given by

$x = 4\cos 3t$

Determine a) position of the particle at $t = 1.25$ sec

b) amplitude and frequency of the particle

Soln:

$x = 4\cos 3t$

At $t = 1.25$ sec, $x = 4\cos\overbrace{(3 \times 1.25)}^{\text{in radians}}$

$= 4\cos 3.75$

$= \cancel{3.99\ m} = -3.28\ m$

$\pi = 180^\circ$

$1 = \frac{180}{\pi}^\circ$

$3.75 = \left(\frac{180}{\pi} \times 3.75\right)^\circ$

Comparing the given function, $x = 4\cos 3t$ with standard expression, $x = r\cos\omega t$

$r = 4$, $\omega = 3$

ie, amplitude $= 4m$, & $\omega = 2\pi N = 3 \Rightarrow N = 3/2\pi = 0.478\ Hz$

12. A body moving with SHM has an amplitude of 1m. and the period of one complete oscillation is 2 sec. what will be the speed ~~of~~ and acceleration of the body 2/5 of a sec after passing the mid position ?.

Tutorial 8

$r = 1\,m$

$T = 2\,sec \Rightarrow 2\pi/\omega = 2 \Rightarrow \omega = 2\pi/2 = \pi$ rad/s

when $t = 2/5$ sec, $y = r \sin \omega t$

$\quad$ ie, $y = 1 \sin(\omega \times 2/5)$ ← radians

$= \sin 2\pi/5$

$= \underline{0.95\,m}$

$V = \omega\sqrt{r^2 - y^2}$

$= \pi\sqrt{1^2 - 0.95^2}$

$= \underline{0.98\ m/s}$

$a = \omega^2 y$

$= \pi^2 \times 0.95$

$= \underline{9.38\ m/s^2}$

or

$V = \frac{dy}{dt} = r\omega \cos \omega t$

$= 1 \times \pi \times \cos \pi \times 2/5$

$= \underline{0.97\ m/s}$

$a = r\omega^2 \sin \omega t$

$= 1 \times \pi^2 \sin \pi \times 2/5$

$= \underline{9.39\ m/s^2}$

13. A point executing SHM has velocities u and v and accelerations a and b in 2 of its positions, to the same side of the mean posn. Show that the distance b/w the 2 positions is $\frac{V^2 - u^2}{a+b}$?.

Tutorial 7

$V = \omega\sqrt{r^2 - y^2}$

ie, $u = \omega\sqrt{r^2 - y_1^2}$

$V = \omega\sqrt{r^2 - y_2^2}$

$V^2 - u^2 = \omega^2\left[(r^2 - y_2^2) - (r^2 - y_1^2)\right]$

$V^2 - u^2 = \omega^2\left[y_1^2 - y_2^2\right] \longrightarrow (1)$

acceler^n, $a = -\omega^2 y$

ie, $a = -\omega^2 y_1$

$b = -\omega^2 y_2$

$$a + b = -\omega^2 (y_1 + y_2) \longrightarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{v^2 - u^2}{a+b} = \frac{\omega^2(y_1^2 - y_2^2)}{-\omega^2(y_1+y_2)} = \frac{(y_1-y_2)(y_1+y_2)}{-(y_1+y_2)}$$

$$= y_2 - y_1$$

= distance b/w 2 position

Hence the proof.

14. A body vibrates in SHM with a period of 5 sec and an amplitude of 1.5 cm. Find the velocity and acceleration of the body:

a) at the mean position

b) at the end of the path

c) at a point 1 cm from the mean position

tutorial (7)

$T = 5$ sec. $\quad r = 1.5$ cm

$= 0.015$ m

$\frac{2\pi}{\omega} = 5$ sec,

$\Rightarrow \omega = \frac{2\pi}{5}$ rad/s

a) at the mean position

$V = r\omega$

$= 0.015 \times \frac{2\pi}{5}$

$= 0.0188$ m/s

$a = \omega^2 y = 0$

b) at the end of the path, ie $y = r$

$V = 0$

$a = \omega^2 y$

$= (2\pi/5)^2 \times 0.015 = 0.024$ m/s^2

c) at a point 1 cm from the mean position,

i.e, $y = 0.01 m$

$V = \omega\sqrt{r^2 - y^2}$

$= \frac{2\pi}{5}\sqrt{0.015^2 - 0.01^2}$

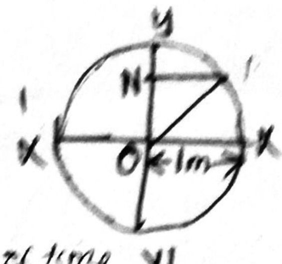
$= \underline{0.014 m/s}$

$a = \omega^2 y$

$= \left(\frac{2\pi}{5}\right)^2 \times 0.01$

$= \underline{0.0158 m/s^2}$

15. A body moving with SHM has an amplitude of 1 m and the period of complete oscillation is 2 sec what will be the velocity and acceleration of the body after 0.4 sec from the extreme position



$r = 1 m$,

$T = 2 sec$

Since ~~time period~~ time period is T sec,

Time from O to y $= \frac{T}{4}$ ($\because$ 1/4 of time period)

$= \frac{2}{4} sec = \underline{0.5 sec}$

Time reqd to travel from O to N

$= 0.5 - 0.4$

$= \underline{0.1 sec}$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \underline{\pi\ rad/s}$ $\Rightarrow y = r \sin \omega t$ ← radian

$= 1 \sin(\pi \times 0.1)$

$= \underline{0.309 m}$

$V = \omega\sqrt{r^2 - y^2}$

$= \pi\sqrt{1^2 - 0.309^2}$

$= \underline{2.99 m/s}$

$a = \omega^2 y = \pi^2 \times 0.309 = \underline{3.05 m/s^2}$

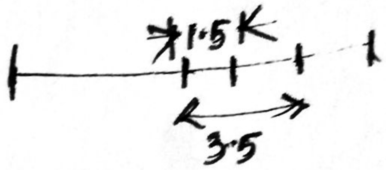
16. A particle moving with SHM has an amplitude of 4·5 m and T is 3·5 sec. Find the time reqd by the particle to pass 2 points which are at a distance of 3·5 m and 1·5 m from the centre and on the same side of the mean position.

Tutorial 6(B)

$r = 4.5\,m$

$T = 3.5\,se$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{3.5} = 1.795\ rad/s$



$y = r \sin \omega t$ radians

ie, $3.5 = 4.5 \sin 1.795 t_1$

Also, $1.5 = 4.5 \sin 1.795 t_2$

ie, $\cancel{\frac{3.5}{1.5} = \frac{\sin 1.795 t_1}{\sin 1.795 t_2}}$

$3.5 = 4.5 \sin \left(1.795 t_1 \times \frac{180}{\pi}\right)$

ie, $0.7777 = \sin 102.845 t_1$

ie, $t_1 = 0.49644\ s$

Similarly,

$0.3333 = \sin \left(1.795 t_2 \times \frac{180}{\pi}\right)$

$0.3333 = \sin 102.845 t_2$

$t_2 = 0.1893\ s$

Time reqd to pass 2 points $= t_1 - t_2$

$= 0.49644 - 0.1893$

$= 0.307144$